# CONSTRUCTION OF SUPERELEMENTS FOR HEAT-CONDUCTION AND POTENTIAL PROBLEMS

Yu. K. Malikov, V. G. Lisienko, and A. E. Vostrotin

UDC 519.6:536.24.02

The use of a superposition of solutions of Poisson's equation is proposed as the system of basis functions for a finite element.

The development of the ideas contained in the finite-element method (FEM) and boundary-element method (BEM) led to the appearance of the hybrid superelement method (SEM) [1, 2]. The system of basis functions for the superelement is constructed using the BEM techniques. A global approximation of the solution within the limits of the superelement often gives better results, even with a smaller number of parameters, then standard finite-element functions of the form in [3]. Another advantage of SEM is that, on combining superelements, two matching conditions (for the potential and the flux) may be written in explicit form; the complete matrix of the system of equations will take the strip structure characteristic of FEM.

BEM is based on the algebraic approximation of integral equations relating the values of the potential T and flux density q at the boundary of the region

$$T_i - \overline{T} = \sum_{j=1}^{M} C_{ij} q_j. \tag{1}$$

The form of the boundary equations for Poisson's equation introduced in [4] is used here

$$-\lambda \Delta T = \tilde{q} \tag{2}$$

with a constant (within the limits of the superelement) source term. As a reference point, the potential in Eq. (1) is taken to be the mean superelement potential

$$\overline{T} = \frac{1}{V} \int_{(V)} T(x,\ y,\ z)\, dV.$$

This choice of reference point is very convenient in solving nonsteady heat-conduction problems by the superelement method [4] and a series of steady potential functions. For example, in the problem of torsion of a prismatic rod [5], $\overline{T}$ determines the rod rigidity. It is more important that the form of the boundary conditions in Eq. (1) allows the matrix $C_{ij}$ to be symmetrized and permits the use of highly effective algorithms for symmetric linear systems in solving the problem.

In determining the influence coefficients $C_{ij}$, the boundary values of T and q are usually approximated by splines of different orders [1-3, 5]. To increase the accuracy of the calculations, it would be desirable to match the approximations of T and q, taking account of the type of equation to be solved and the geometry of the region. Regrettably, this matching cannot be accomplished directly, using existing BEM techniques. However, the principle of superposition of the solutions of a linear equation (for example, the Poisson's equation) allows the influence coefficients found using BEM to be transformed so as to ensure self-consistent approximation of the boundary values of T and q. In the present work, it is shown that this also permits significant reduction in the number of parameters required for global approximation of the solution within the limits of the superelement. The superelements constructed in this way are said to be regular.

S. M. Kirov Ural Polytechnic Institute, Sverdlovsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 55, No. 6, pp. 1020-1027, December, 1988. Original article submitted June 18, 1987.

 0022-0841/88/5506-1444$12.50 

TABLE 1. Coefficients of the Generalized Function in Eq. (6)

| Function | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|
| $E_0$ | 0 | $-\operatorname{sign} v_i$ | 0 |
| $E_1$ | $-v_i/F$ | $-\operatorname{sign} v_i\, u_i/F$ | 0 |
| $E_2$ | $-2u_i v_i/F^2$ | $-\operatorname{sign} v_i\,(u_i^2-v_i^2)/F^2$ | $-v_i/F$ |
| $G_0$ | $u_i$ | $-\lvert v_i\rvert$ | $F(1-\ln R_2)$ |
| $G_1$ | $-u_i v_i/2$ | $\operatorname{sign} v_i\, v_i^2/2$ | $Fv_i(2\ln R_2-3)/4$ |

## Zero-Order Regular Elements

Consider the region with a boundary consisting of N rectilinear segments (Fig. 1a). The heat flux at the boundary of the region is approximated by a step function taking the constant value $q_i$ on the i-th segment of the contour. At the boundary of the region, N control points $(X_i, Y_i)$ are chosen, and the temperature at the i-th control point is denoted by $T_i = T(X_i, Y_i)$. The position of the control points at the boundary of the region may be arbitrary. In many cases, it is more convenient to consider the mean temperature value $\bar{T}_i$ at the i-th segment of the contour rather than the temperatures at the control points.

Let T(X, Y, j) be the solution of Eq. (2) when $\tilde{q} = -q_j F_j/V$, with the following conditions at the boundary

$$-\lambda\frac{\partial T(X, Y, j)}{\partial n} = \begin{cases} q_j \text{ on the j-th segment of the boundary} \\ 0 \text{ on other segments,} \end{cases}$$

where n is the internal normal. The values of $T_i$, $\bar{T}_i$, $\bar{T}$ corresponding to this temperature distribution are denoted by $T_i(j)$, $\bar{T}_i(j)$, $\bar{T}(j)$, and the coefficients $C_{ij}$ in Eq. (1) are defined as follows (M = N)

$$C_{ij} = (T_i(j) - \bar{T}(j))/q_j;\quad \bar{C}_{ij} = (\bar{T}_i(j) - \bar{T}(j))/q_j. \tag{3}$$

It follows from the principle of superposition of the solutions of Eq. (2) that the coefficients in Eq. (3) ensure accurate satisfaction of the boundary Eq. (1) for any function T(X, Y) which corresponds to Eq. (2) and has piecewise-constant values of the normal derivative at the boundary of the region.

Three-dimensional zero-order regular elements may be constructed similarly. It is especially simple to determine the coefficients $\bar{C}_{ij}$ for prismatic bodies with a constant form of the cross section. Consider, for example, a right prism of height h. The lateral faces of the prism are numbered 1, 2, ..., N -2, and the numbers N - 1, N are assigned to its upper and lower bases. Then the matrix of coefficients $\bar{C}_{ij}$ takes the form

$$\bar{C}_{ij} = \bar{C}'_{ij},\quad i, j = 1, 2, \ldots, N-2;$$

$$\bar{C}_{ij} = \bar{C}_{ji} = 0,\quad i = N-1, N,\ j = 1, 2, \ldots, N-2;$$

$$\bar{C}_{ii} = h/3\lambda,\quad i = N-1, N;\quad \bar{C}_{N-1, N} = \bar{C}_{N, N-1} = -h/6\lambda.$$

Here $\bar{C}'_{ij}$ is the matrix of coefficients of dimensionality (N-2) × (N-2) for a two-dimensional regular element coinciding in form with the base of the prism.

## First-Order Regular Elements

Passing to a linear approximation of the flux at the boundary of the region, it must be taken into account that q usually undergoes a discontinuity at the ends of the rectilinear boundary elements. Therefore, for each segment i bounding the region (Fig. 1a), two control points must be introduced for the heat flux; they are assigned the numbers i and i' = N + i (Fig. 1b). Thus, the heat-flux density at the i-th segment will vary linearly from $q = q_i$ to $q = q_i'$; in the general case, the value of q at point i' of segment i may not coincide with

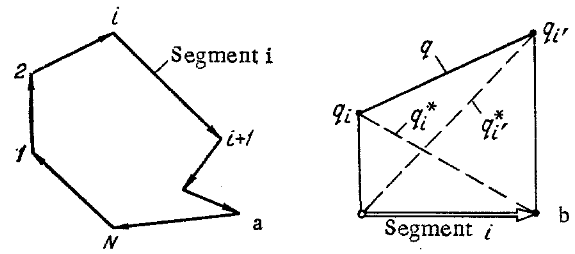


Fig. 1. Numbering of boundary elements: a) zero-order approximation of flux; b) first-order case.

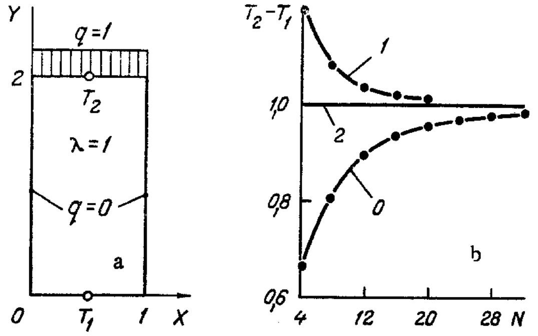


Fig. 2. Test example: a) geometric region; b) BEM solution with zero-order approximation of flux. The figures on the curves give the order of approximation of the potential.

the value of q at point i + 1 of segment i + 1: $q_{i'} \neq q_{i+1}$. Accordingly, 2N trial functions $q_i^*$, $q_i^{*\prime}$ (i = 1, 2, ..., N; i' = N + 1, N + 2, ..., 2N), each of which is nonzero only on the i-th segment, are defined. Within the limits of the segment, the trial functions vary linearly, as in Fig. 1b.

Suppose that T(X, Y, j) is the solution of Eq. (2) when $\tilde{q} = -q_j F_j/2V$ with the boundary conditions

$$-\lambda \frac{\partial T(X, Y, j)}{\partial n} = q_j^*, \; j = 1, 2, \ldots, 2N.$$

At the boundary of the region, 2N control points are chosen, and $(2N)^2$ coefficients $C_{ij}$ are defined in Eq. (1) (M = 2N)

$$C_{ij} = (T_i(j) - \bar{T}(j))/q_j. \tag{4}$$

It follows from the principle of superposition of solutions of Eq. (2) that the coefficients in Eq. (4) ensure accurate satisfaction of Eq. (1) for any function T(X, Y) which corresponds to Eq. (2) and has a piecewise-linear normal derivative at the boundary of the region.

Regular superelements of second and higher order may be constructed similarly.

## Comparison with BEM Method

Consider the simplest heat-conduction problem for a rectangular region (Fig. 2a). The heat flux at the upper side of the rectangle q = 1 is compensated by the uniformly distributed heat sources $\tilde{q} = -0.5$. In these conditions, the function $T = 0.25Y^2$ gives a solution of Eq. (2). The temperature difference between the upper and lower sides of the rectangle is $T_2-T_1 = 1$.

The value of $T_2-T_1$ is now determined by direct BEM, which was exhaustively described in [2]. If four constant rectilinear boundary elements coinciding with the sides of the rectangle are adopted, it is found that $T_2 - T_1 = 0.66$. Using four elements which are linear with respect to the potential and constant with respect to the flux, it is found that $T_2 - T_1 = 1.21$. Finally, 4K boundary elements may be adopted, dividing each side of the rectangle

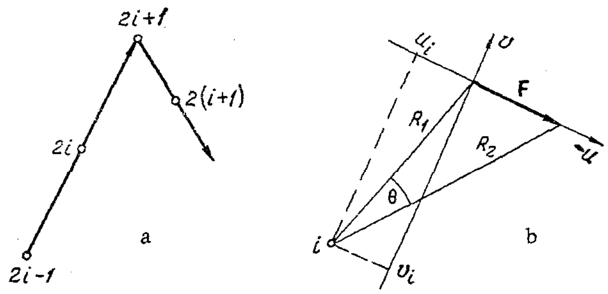


Fig. 3. Calculation of the coefficients: a) numbering of the nodal points; b) local system of coordinates.

into K identical parts. As is evident from Fig. 2b, the rate of convergence of BEM (taking account of the simplicity of the problem) leaves room for improvement. The accurate value of $T_2 - T_1$ for the given example may be obtained using four boundary elements with a parabolic approximation of the temperature at the boundary of the region.

A set of other examples may be given, showing that the piecewise-constant approximation of the flux at the boundary of the region corresponds roughly to the quadratic approximation of the temperature (potential). The system of BEM equations for N boundary elements here consists of 2N equations with $6N^2$ different coefficients. In the class of functions with piecewise-constant values of the derivative at the boundary of the region, 2N BEM equations are approximately equivalent to N boundary equations for a regular element with $N^2$ coefficients $C_{ij}$.

The reduction in the number of equations and coefficients has a favorable influence on the stability of the calculations relative to rounding error, and also considerably simplifies the solution of the whole problem for the complex region consisting of many superelements.

## Calculation of Coefficients

Determining the coefficients $C_{ij}$ reduces to solving Eq. (2) in a specified region with special boundary conditions of the type considered above. In [4], the finite-difference method was used for this purpose. However, this method is not very effective and is not readily susceptible to automation. At the same time, the BEM technique allows a better approximation to the coefficients $C_{ij}$ to be obtained using simple analytical expressions. Below, relations are given allowing the coefficients $\tilde{C}_{ij}$ to be determined for zero-order two-dimensional regular superelements. The choice of these relations is based on the results of [2], and is not given here.

Consider the region in Fig. 1a, and let the centers of the segments which form its boundary be numbered by the even indices 2i, i = 1, 2, ..., N, proceeding clockwise around the contour (Fig. 3a). The dimensions of the segments $F_{2i}$ and the piecewise-constant values of the flux density at the boundary of the region $q_{2i}$ are also assigned these indices. The ends of the segments are numbered by the odd numbers 1, 3, 5, ..., 2N − 1, so that the vector $F_{2i}$ connects the points 2i − 1, 2i + 1. On each segment, the temperature is approximated by a parabola, taking the values $T_j$, j = 2i − 1, 2i, 2i + 1 at three control points of the segment.

Taking account of the condition $-\tilde{q}V = \sum_{j=1}^{N} F_{2j}q_{2j}$, the equations of direct BEM for Eq. (2) may be written in the form

$$\sum_{j=1}^{2N} E_{ij}T_j = -\frac{1}{\lambda}\sum_{j=1}^{N} G_{ij}q_{2j}, \quad i = 1,\ 2,\ \ldots,\ 2N. \tag{5}$$

It is expedient to introduce the generalized function Φ(i, F) depending on the position of the point i relative to the vector F, associated with the local coordinate system uOv (Fig. 3b):

$$\Phi(i,\ \mathbf{F}) = [A_1 \ln (R_2/R_1) + A_2\theta + A_3]/2\pi. \tag{6}$$

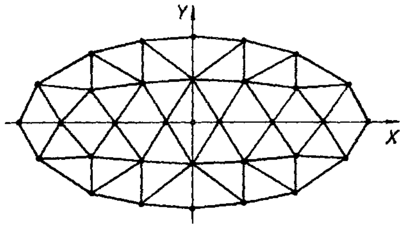


Fig. 4. Discretization of elliptical rod cross section using finite and boundary elements.

With specific values of the coefficients $A_1$, $A_2$, $A_3$, Eq. (6) defines the functions $E_0$, $E_1$, $E_2$, $G_0$, $G_1$ (Table 1). Using these functions, the coefficients in Eq. (5) for i = 1, 2, ..., 2N and different j may be found

$$E_{ij} = 4E_1(i, \mathbf{F}_j) - 4E_2(i, \mathbf{F}_j), \quad j = 2, 4, 6, \ldots, 2N;$$
$$E_{ij} = E_0(i, \mathbf{F}_{j+1}) - 3E_1(i, \mathbf{F}_{j+1}) + 2E_2(i, F_{j+1}) -$$
$$- E_1(i, F_{j-1}) + 2E_2(i, F_{j-1}), \quad j = 1, 3, 5, \ldots, 2N-1;$$
$$G_{ij} = G_0(i, \mathbf{F}_j) - \frac{F_j}{V} \sum_{K=1}^{N} G_1(i, \mathbf{F}_{2k}), \quad j = 2, 4, 6, \ldots, 2N.$$

The coefficients of the matrices $E_{ij}$ on the left-hand side of Eq. (5) satisfy the condition $\sum_{j=1}^{2N} E_{ij} = 0$, which is used to determine the diagonal terms $E_{ii}$, i = 1, 2, ..., 2N. Consequently, one of the equations in Eq. (5) is a consequence of the others, and Eq. (5) may be rewritten in the form

$$\sum_{j=1}^{2N-1} E_{ij}(T_i - T_{2N}) = -\frac{1}{\lambda} \sum_{j=1}^{N} G_{ij} q_{2j}, \quad i = 1, 2, \ldots, 2N-1, \tag{7}$$

where the temperature $T_{2N}$ at the center of the next segment plays the role of reference point. Then the matrices $E_{ij}$ of dimensions (2N − 1) × (2N − 1) may be manipulated, and Eq. (7) takes the form

$$T_i - T_{2N} = \frac{1}{\lambda} \sum_{j=1}^{N} B_{ij} q_{2j}, \quad i = 1, 2, \ldots, 2N-1. \tag{8}$$

Conversion from the temperature values at the control points $T_i$, i = 1, 2, ..., 2N to the mean temperature of the segments $\overline{T}_{2i}$, i = 1, 2, ..., N, is undertaken by the formula $\overline{T}_{2i} = (T_{2i-1} + 4T_{2i} + T_{2i+1})/6$. Adding Eq. (8) with the corresponding weighting factors, it is found that

$$\overline{T}_{2i} - T_{2N} = -\frac{1}{\lambda} \sum_{j=1}^{N} \overline{B}_{ij} q_{2j}, \quad i = 1, 2, \ldots, N. \tag{9}$$

The system in Eq. (9) differs from Eq. (1) in that the temperature $T_{2N}$ rather than the mean temperature of the superelement $\overline{T}$ is taken as the reference point here. The relation between $\overline{T}$ and the values of the functions at the boundary of the region is given by the second Green's formula. Using the weighting function $R^2 = X^2 + Y^2$, it is simple to establish that, for the given class of regions with a piecewise rectilinear boundary and the class of functions with a piecewise constant normal derivative at the boundary, the following relation holds

$$\overline{T} - T_{2N} = \sum_{i=1}^{N} \left[ \alpha_i (\overline{T}_{2i} - T_{2N}) - \frac{1}{\lambda} \beta_i q_{2i} \right], \tag{10}$$

where

$$\alpha_i = F_{2i}(-X_{2i}\sin\psi_{2i} + Y_{2i}\cos\psi_{2i})/2V;$$

$$\beta_i = F_{2i}[X_{2i-1}^2 + X_{2i-1}X_{2i+1} + X_{2i+1}^2 + 3\bar{R}^2 + Y_{2i-1}^2 + Y_{2i-1}Y_{2i+1} + Y_{2i+1}^2]/12V;$$

$$\bar{R}^2 = \sum_{i=1}^{N}[(X_{2i+1} - X_{2i-1})(Y_{2i+1} + Y_{2i-1})(Y_{2i+1}^2 + Y_{2i-1}^2) -$$

$$-(Y_{2i+1} - Y_{2i-1})(X_{2i+1} + X_{2i-1})(X_{2i+1}^2 + X_{2i-1}^2)]/12V.$$

Using Eq. (9), $\bar{T}_{2i}$ may be eliminated from Eq. (10)

$$\bar{T} - T_{2N} = -\frac{1}{\lambda}\sum_{i=1}^{N}\gamma_i q_{2i}, \tag{11}$$

where

$$\gamma_j = \beta_j + \sum_{i=1}^{N}\alpha_i\bar{B}_{ij}.$$

Subtracting Eq. (11) from each relation in Eq. (9), this system is transformed to the form in Eq. (1)

$$\bar{T}_{2i} - \bar{T} = \sum_{j=1}^{N}\bar{C}_{ij}q_{2j}, \quad i = 1,\ 2,\ \ldots,\ N, \tag{12}$$

where $\bar{C}_{ij} = (\gamma_i - \bar{B}_{ij})/\lambda$.

The above analytical Eqs. (6)-(12) are very amenable to computer programming and allow the coefficients $\bar{C}_{ij}$ and the solution of Eq. (2) to be very rapidly found. Calculation for a region of arbitrary form with a contour consisting of 100 segments requires around 15 sec of machine time when using an EC-1030 computer.

## Example of Use

It was shown in [4] that the regular-element method (REM) permits effective solution of many problems of nonsteady heat conduction associated with modeling of the complex heat transfer in different units. The use of REM in solving the problem of the heating of regular bodies in a counterflow furnace permits severalfold reduction in volume of the calculations in comparison with the finite-difference method. More considerable benefits are obtained for bodies of complex form, such as wide-band girders, rails, and I beams.

However, the efficiency of the method in solving complex problems depends on many factors which are not readily taken into account and hinder quantitative estimation. Therefore, it is more important to consider simple test examples, for which it may be established where the limits of applicability of a particular method are reached. One such example was considered above. Another is associated with the torsion of a rod of constant elliptical cross section, with major semiaxis of length 4 and minor semiaxis 2 (Fig. 4). The rigidity of the rod in torsion is determined by the integral

$$J = 2\int_{(V)} T(X,\ Y)\,dXdY = 2V\bar{T},$$

where the function T(X, Y) is the solution of Eq. (2) with $\lambda = 1$, $\tilde{q} = -2$. At the boundary of the region, T(X, Y) = 0.

This problem was solved in [1] using FEM (48 linear finite elements) and BEM (16 linear boundary elements). The results obtained (J = 4.560 for FEM and J = 4.487 for BEM) differ by approximately 10% from the accurate solution for an ellipse (J = 5.026) and by 5% for the inscribed 16-sided figure (J = 4.785). Using REM — Eqs. (6)-(12) — the result obtained is

J = 4.793, and the corresponding errors are 4.8 and 0.2%. Thus, the REM error in the given case is entirely determined by the error in describing the geometry of the region using a set of rectilinear segments. Using 32 rectilinear boundary elements, for example, the result obtained is J = 4.966, which practically coincides with the accurate value for an inscribed 32-sided figure, but differs from the solution for an ellipse by 1.2%.

This example shows that linear or quadratic approximation of the flux at a rectilinear boundary element is not particularly meaningful in many cases, whereas quadratic approximation of the potential almost always gives benefits in terms of the accuracy of the calculations. Another conclusion is that, with the aim of improving the description of the body geometry, it is desirable to use curvilinear elements with a high order of approximation. The difficulties arising here are associcated with the lack of corresponding analytical solutions, which means that it is necessary to use quadratic Gaussian formulas in calculating the influence coefficients, with considerable increase in the consumption of machine time.

## NOTATION

$\lambda$, thermal conductivity; T, potential; q, q*, flux density; $\tilde{q}$, internal source; F, dimension of boundary element; V, dimension of superelement; X, Y and u, v, global and local coordinate systems.

## LITERATURE CITED


1. K. Brebbiya and S. Uoker, Application of Boundary-Element Method in Engineering [Russian translation], Moscow (1982).
2. P. Benerdzhi and R. Butterfield, Boundary-Element Method in Applied Sciences [Russian translation], Moscow (1984).
3. O. Zenkevich and K. Morgan, Finite Elements and Approximations [in Russian], Moscow (1986).
4. Yu. K. Malikov, Inzh.-Fiz. Zh., 50, No. 4, 790 (1986).
5. S. Krauch and A. Starfild, Boundary-Element Method in the Mechanics of Solid Bodies [Russian translation], Moscow (1987).